# GEOMETRIC CORRELATIONS IN REAL MULTIPLEX NETWORKS

Kaj Kolja KLEINEBERG | ✉ • kkl@ffn.ub.edu
• @KoljaKleineberg
Universitat de Barcelona

Marian BOGUNA | ✉ • marian.boguna@ub.edu
Universitat de Barcelona

M. Angeles SERRANO | ✉ • marian.serrano@ub.edu
Universitat de Barcelona

Fragkiskos PAPADOPOULOS | ✉ • f.papadopoulos@cut.ac.cy
Cyprus University of Technology

# **The problem of routing: efficient forwarding** of a message from a source to a target


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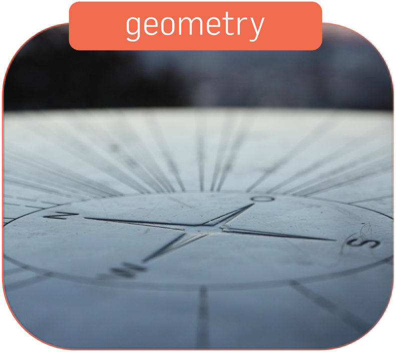
**Diffusion** is **not an efficient way** to perform navigation.

# A map of the system reveals underlying geometry

and provides notion of distance and direction

# A map of the system reveals underlying geometry

and provides notion of distance and direction

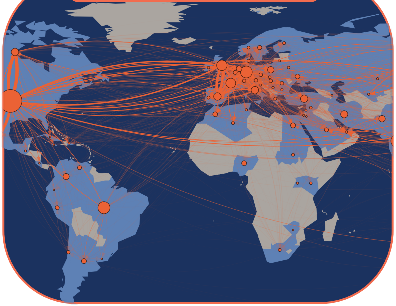
map

geometry

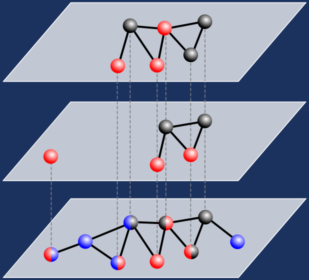
Underlying **geometry** allows **efficient routing** in networks with only **local knowledge.**

# In reality networks form interacting entities
in larger and more complex systems

interacting



multiplex



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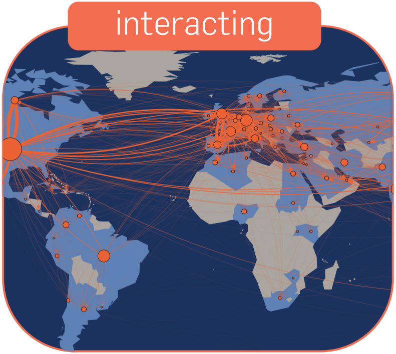
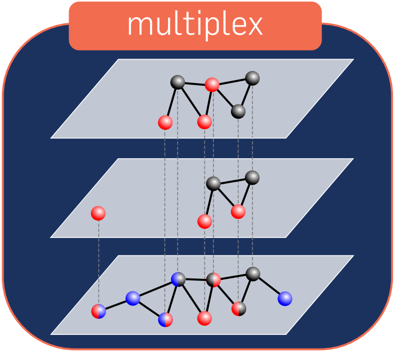



Do **more interacting network layers** improve the performance of **routing?**

# **Geometric correlations** in real multiplex networks

**Geometric correlations**
in real multiplex networks

**Mutual greedy routing**
and geometric correlations

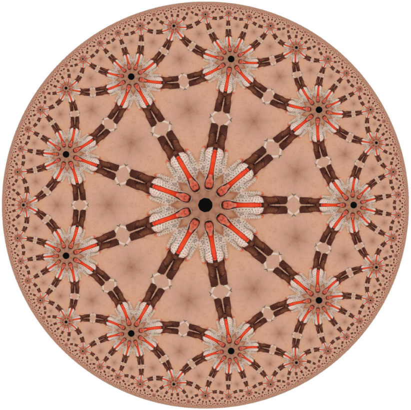
**Geometric correlations**
in real multiplex networks

**Mutual greedy routing**
and geometric correlations

**Internet multiplex**
correlations and routing

# Geometric correlations in real multiplex networks

# The map: real complex networks obey hyperbolic geometry rather than Euclidean



$$x_{ij} \approx r_i + r_j + \ln \frac{\Delta\theta_{ij}}{2}$$

Image taken from Nature Communications 1, 62 (2010)

# **Nodes optimize the product of** popularity and similarity

**Popularity:** Birth time $t = 1, 2, 3, \ldots$

**Similarity:** Position on a circle given by angular coordinate $\theta$

**Growing network:** New node $t$ is placed randomly on the circle and connects to $m$ existing nodes $s$ that minimize the product of popularity times similarity

$$s \times \Delta\theta_{st}$$

Nature 489, 537–540 (2012)

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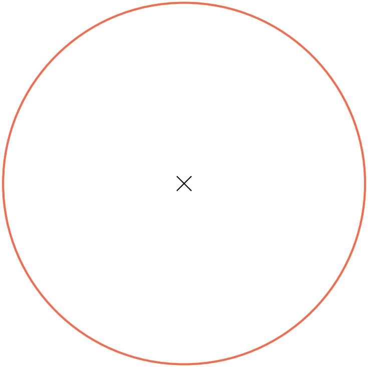

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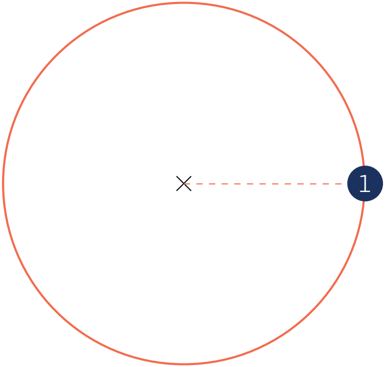

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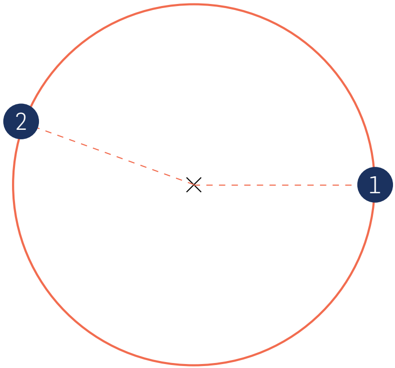

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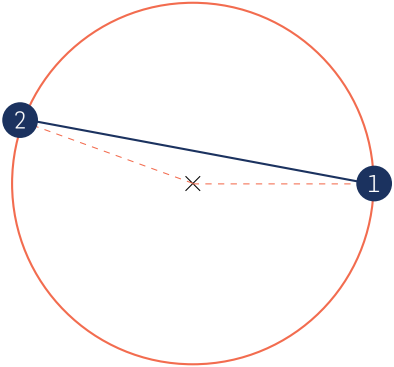

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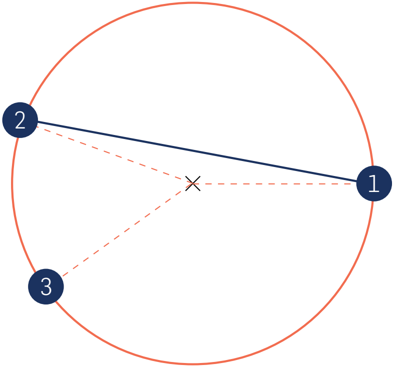

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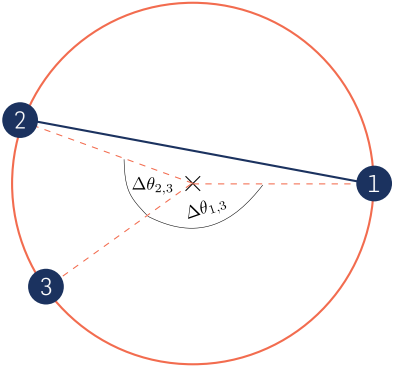

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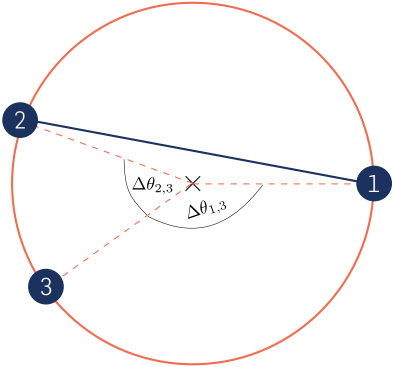


$$2 \times \Delta\theta_{2,3} < 1 \times \Delta\theta_{1,3}$$

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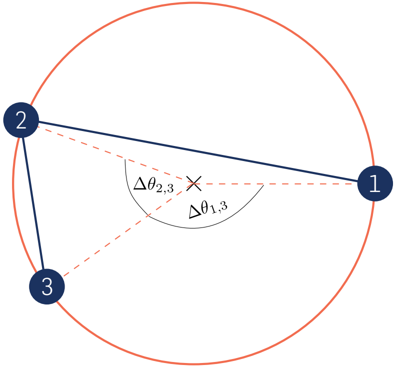


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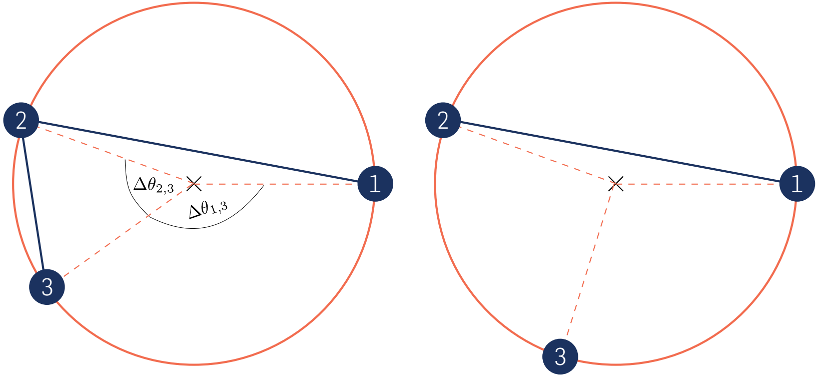


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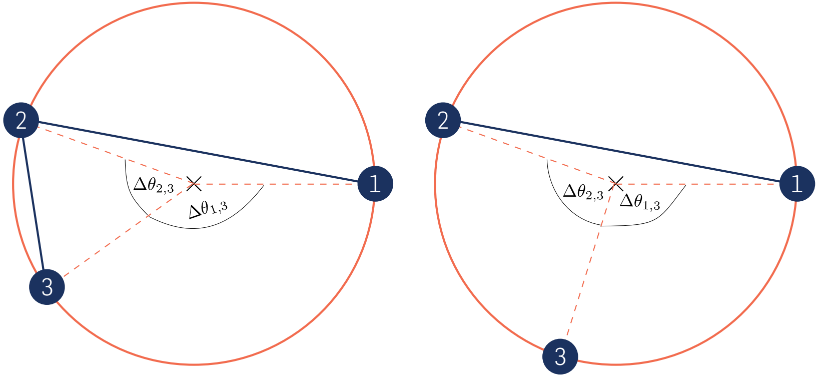


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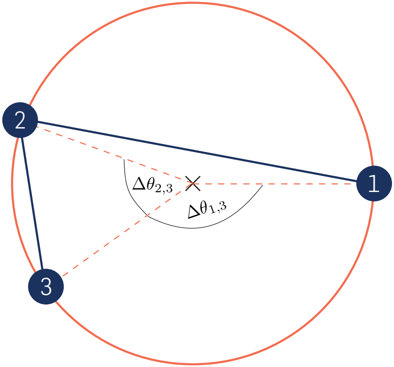


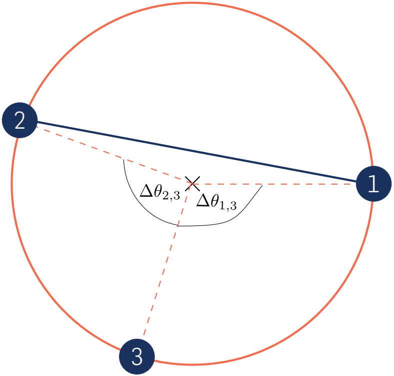


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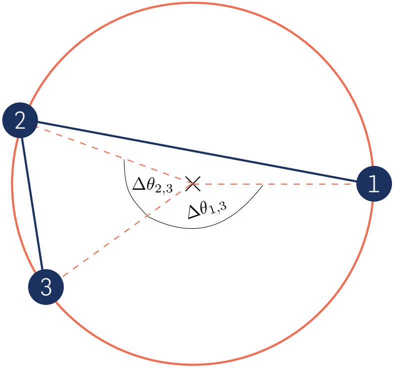


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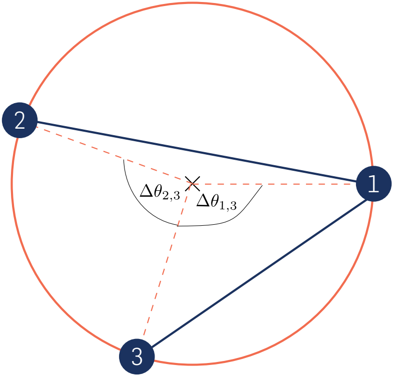


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# Popularity and similarity optimization
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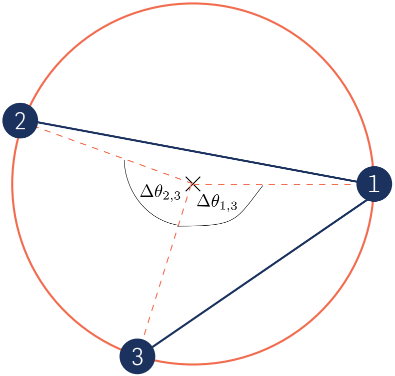
Radial coordinate: $r_t = \ln t$



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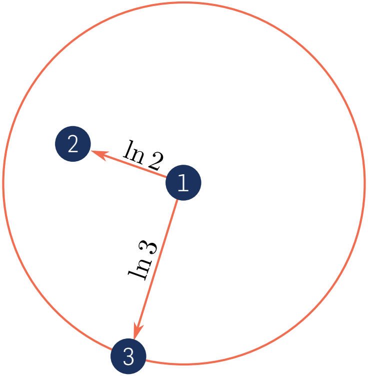

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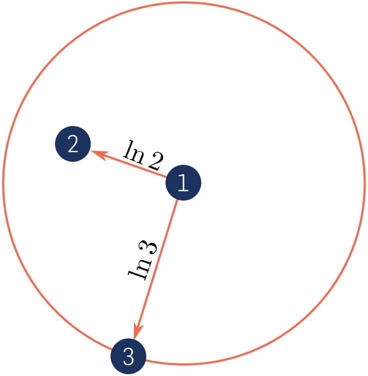

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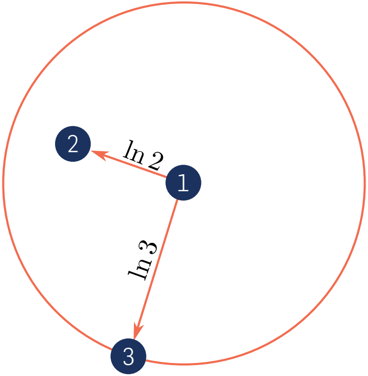

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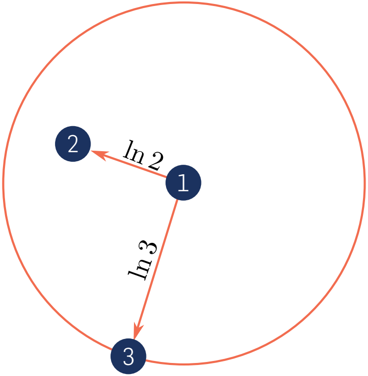

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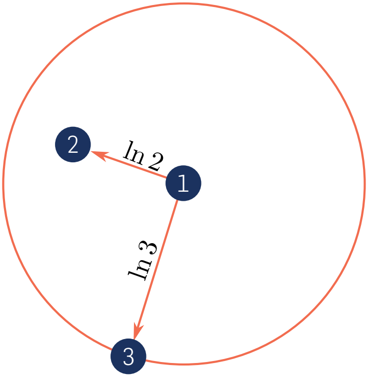

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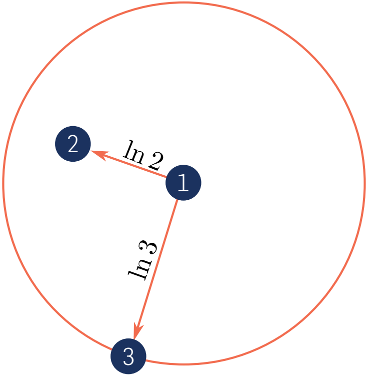


**Generalization:** Nodes $i$ and $j$ are connected with probability

$$p(x_{ij}) = \frac{1}{1 + e^{1/(2T)(x_{ij}-R)}} \qquad T : \text{Temperature}$$

# **Real complex networks can be embedded** in hyperbolic space

**Idea:** Invert hyperbolic network model

**Maximum likelihood:** Find node coordinates that maximize probability to reproduce the observed topology with the model

**Details:** PRE 92, 022807 (2015)

# **Constituent network layers of real multiplex systems**
are embedded into separate hyperbolic spaces

**Internet**
IPv4 and IPv6
protocol

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transportation in India

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**arXiv**
co-authorship in different categories

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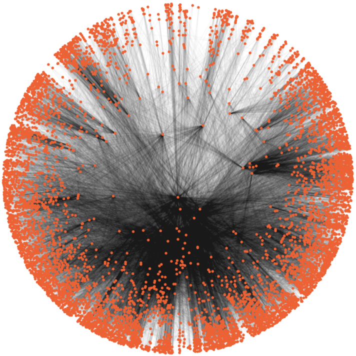
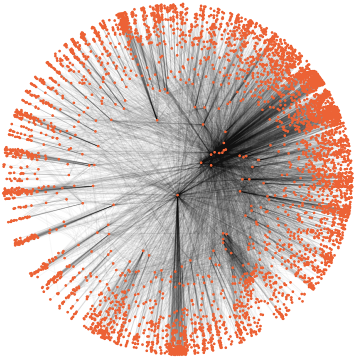
Internet IPv4 network



Internet IPv6 network



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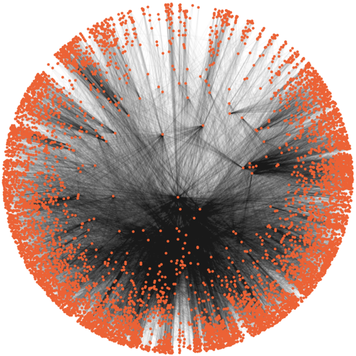
## Internet IPv4 network



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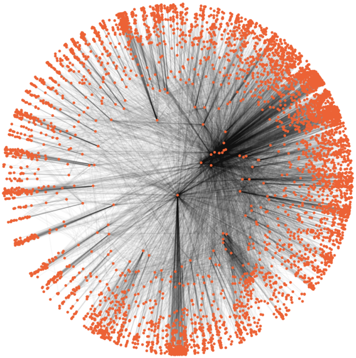
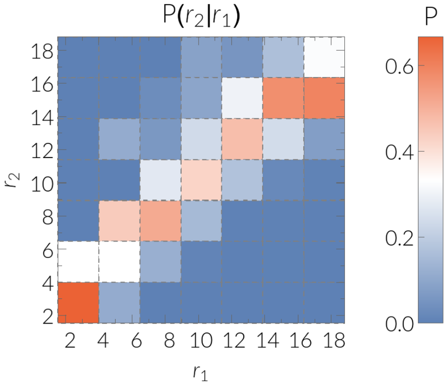


Are **coordinates** of same nodes in different layers **correlated?**

# **Radial coordinates are strongly correlated**
between different layers



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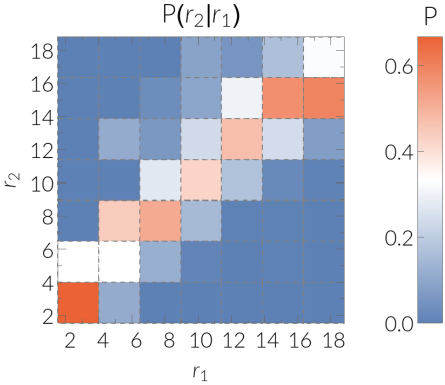
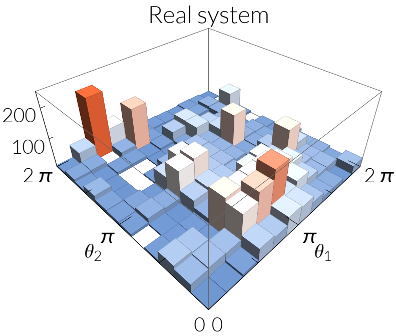



**Radial correlations** are equivalent to **degree degree correlations** found in many studies.

# **Node clusters similar in both layers are overabundant** in real compared systems to reshuffled counterparts



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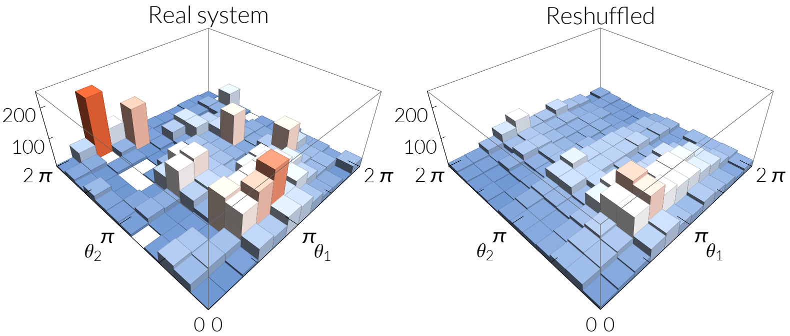

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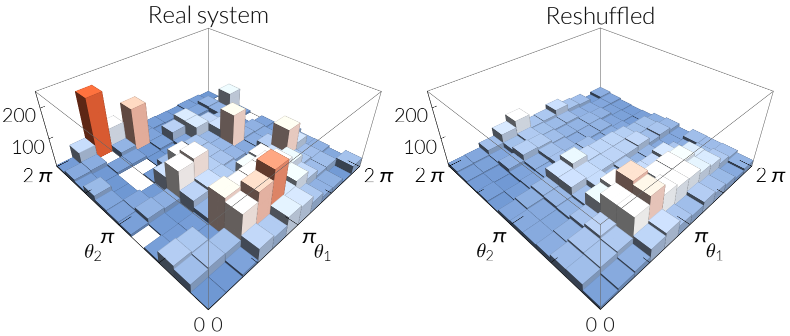
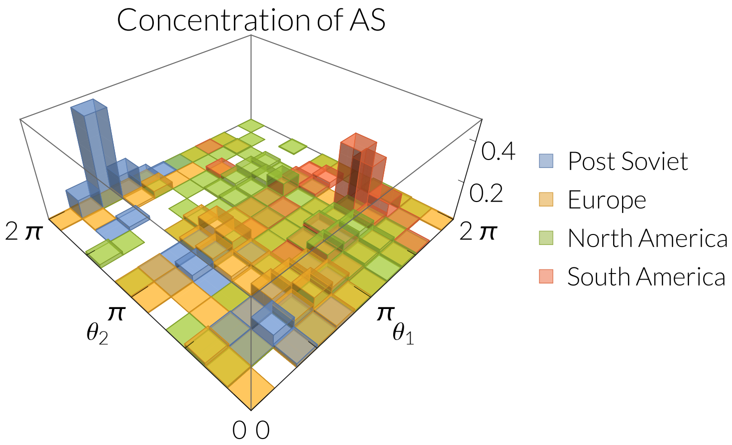
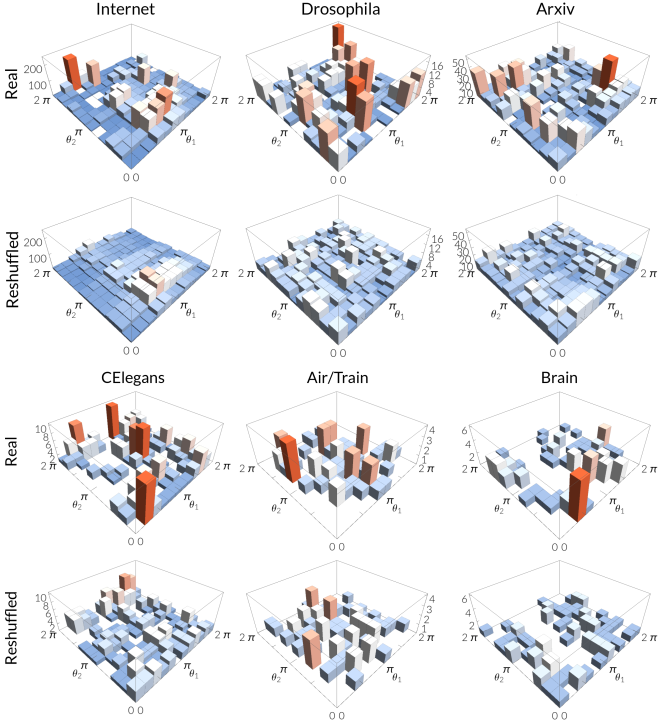



**Angular correlations** exist and give rise to **multidimensional communities.**

# Generalized communities in the Internet

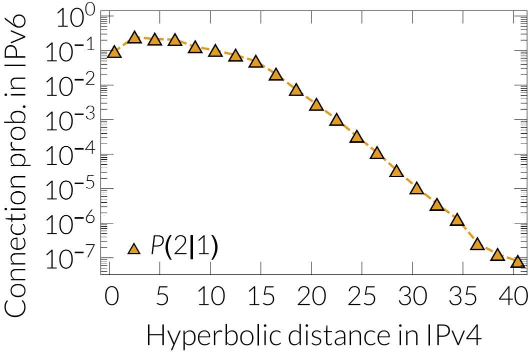
belong to certain geographic regions





Internet
Drosophila
Arxiv
Real
Reshuffled
CElegans
Air/Train
Brain
Real
Reshuffled

# Distance between pairs of nodes in one layer is an indicator of the connection probability in another layer



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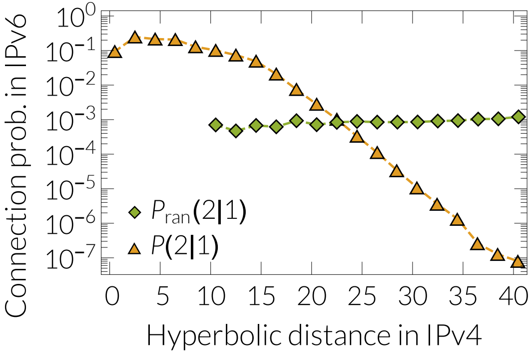

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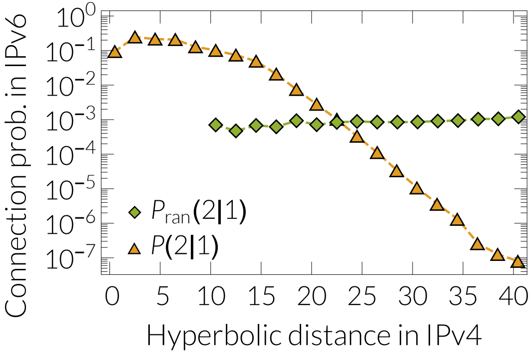
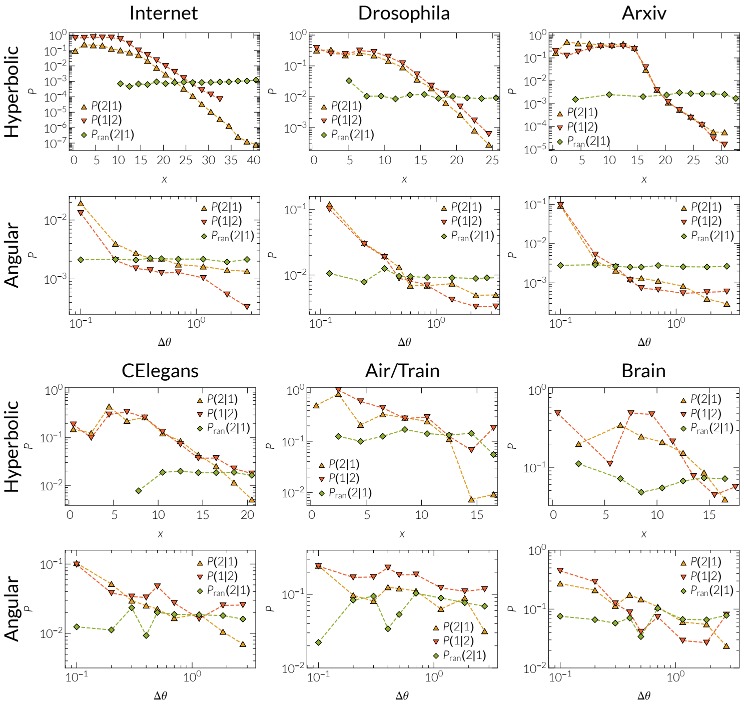



**Geometric correlations** enable precise **trans-layer link prediction.**



Internet
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Hyperbolic
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P
x
Δθ
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# Geometric correlations exist in real multiplex systems
and generalize community detection and link prediction

**Metric correlations**
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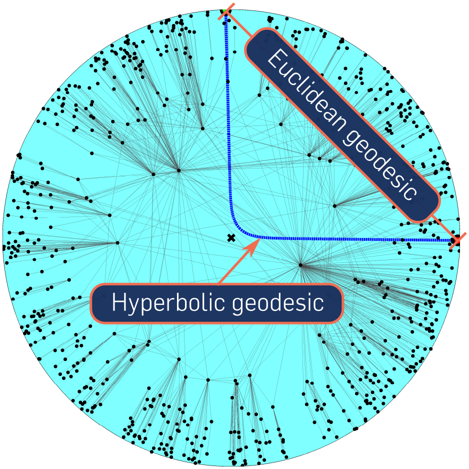
# Mutual greedy routing

# **Greedy routing in single network using hyperbolic space** allows efficient navigation relying only on local knowledge

$$R > d_{ij} = \cosh^{-1}\left[\cosh r_i \cosh r_j - \sinh r_i \sinh r_j \cos \Delta\theta_{ij}\right]$$



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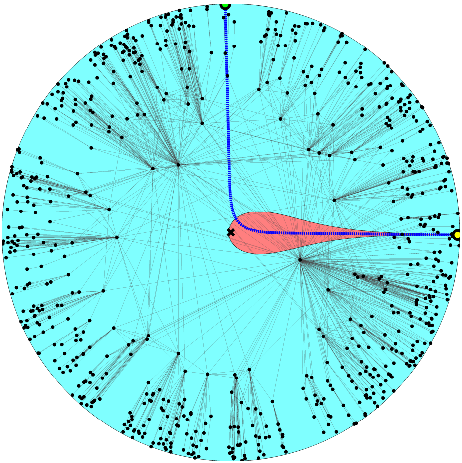

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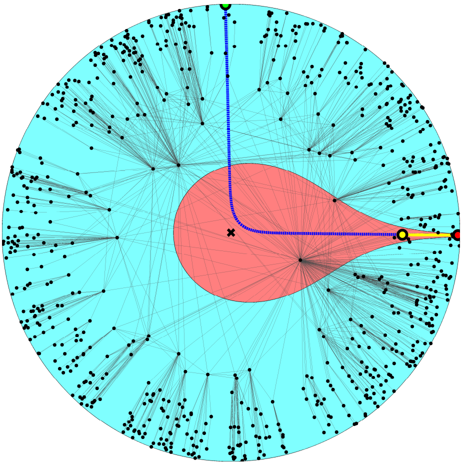

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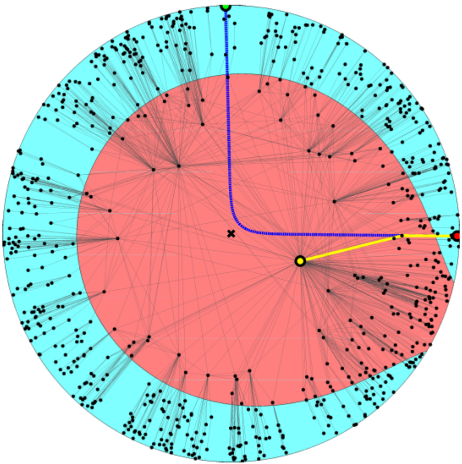

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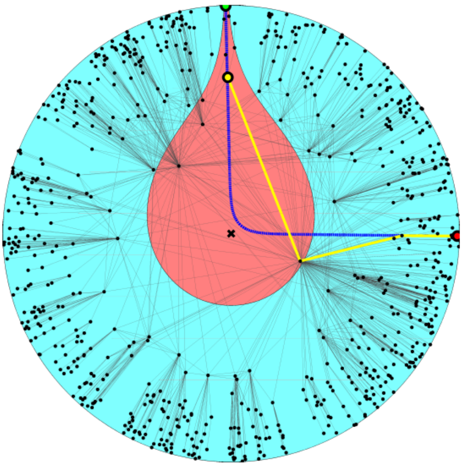

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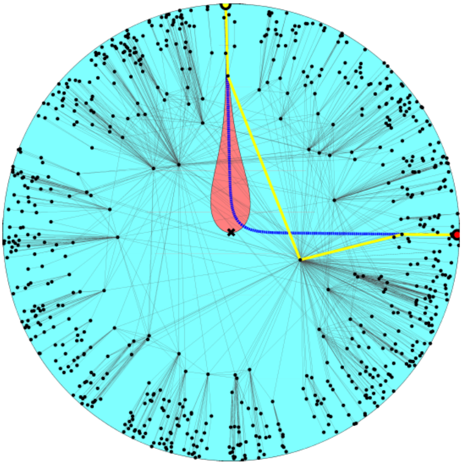
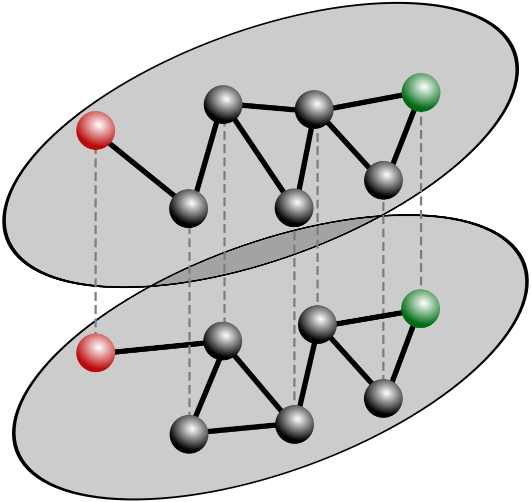


# Mutual greedy routing uses multiple layers simultaneously:
dependence on geometric correlations and number of layers?



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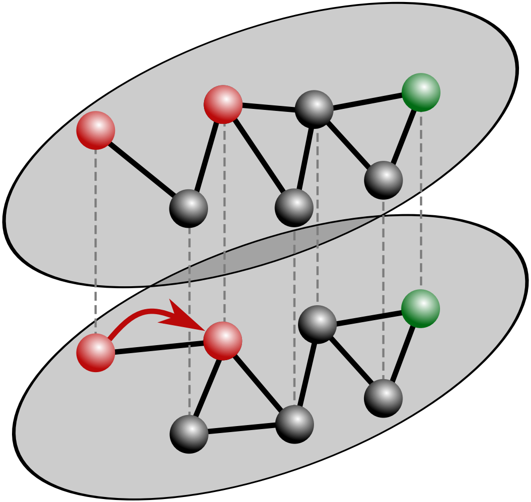

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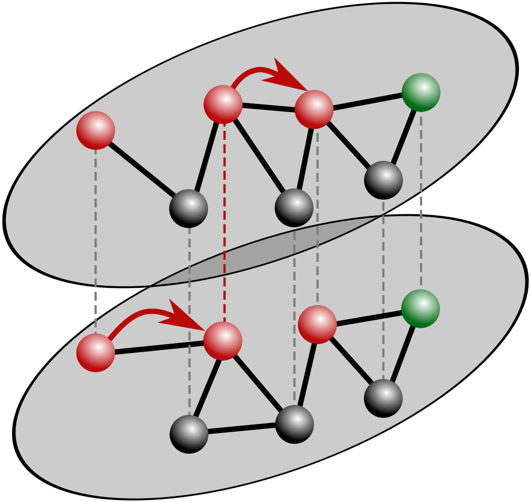

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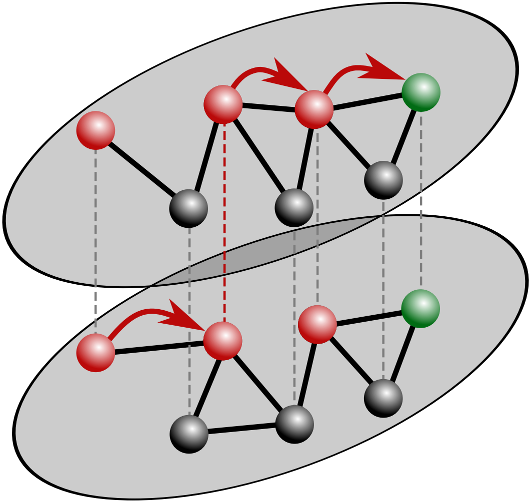

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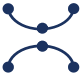

**Hyperbolic routing**
forwarding to the neighbor with shortest hyperbolic distance to target in any of the layers

**Angular routing**
forwarding to the neighbor with shortest angular distance to target in any of the layers

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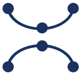
Need for model to **vary correlations independently** from layer topology to study impact of correlations.

# **Correlated metric multiplex model allows to tune** geometric correlations independently from layer topologies



**Constituent layer**
topologies according to hyperbolic model

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topologies according to hyperbolic model

**Geometric correlations**
tuned independently from constituent layer topologies

**Radial correlations**
Gumbel-Hougaard copula controlled by $\nu \in [0, 1]$

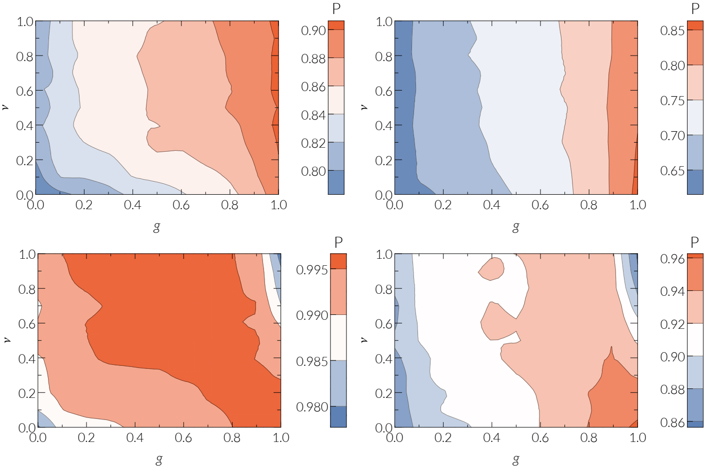
**Angular correlations**
truncated Gaussian distribution controlled by $g \in [0, 1]$

# Correlations improve performance mutual greedy routing
using angular or hyperbolic distances

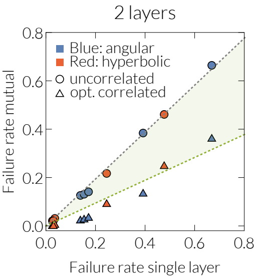
**Hyperbolic** **Angular**

**T = 0.8**

**T = 0.1**



# **Geometric correlations determine the improvement of** mutual greedy routing by increasing the number of layers



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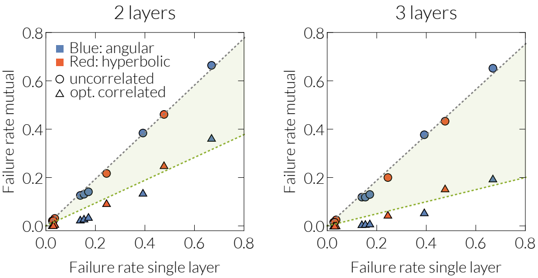

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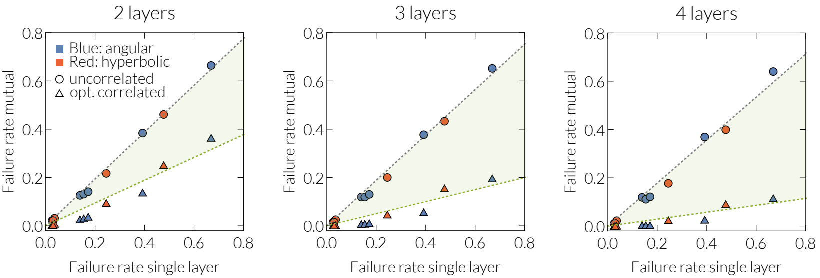

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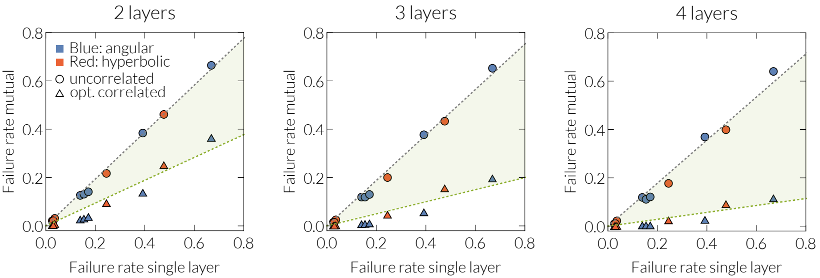


Constant **failure mitigation factor** as inverse of the slope for **optimal and uncorrelated case.**

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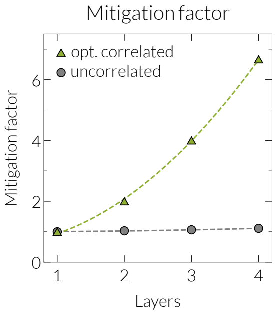

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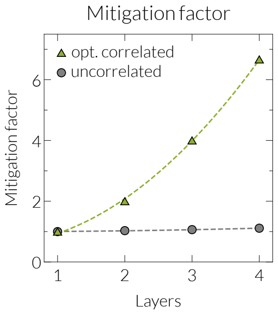


**Additional layers** make system **perfectly navigable** if **correlations** are present, but **otherwise are useless.**

# Metric correlations increase the performance
of mutual greedy routing

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# **Metric correlations increase the performance** of mutual greedy routing

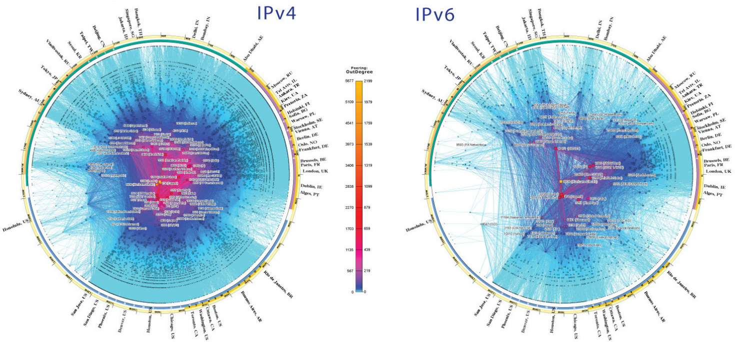
**Geometric correlations** improve mutual greedy routing

**Uncorrelated layers** do not improve mutual navigability

**Optimal correlations** make system perfectly navigable

# The IPv4 IPv6 Internet multiplex

# **Constituent layers of the Internet multiplex**
have significantly different sizes






$N = 40000$ $\qquad$ $N = 5000$

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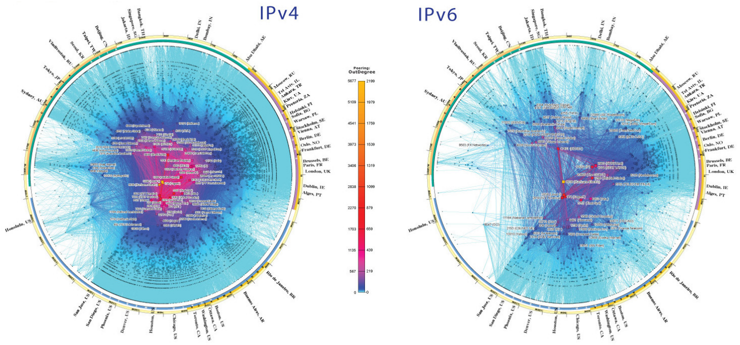
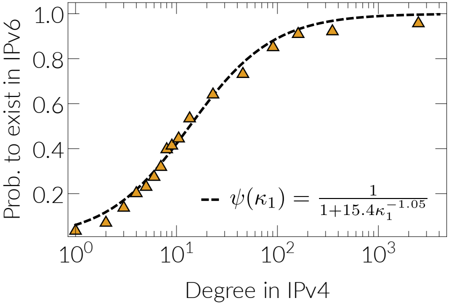





$N = 40000$ $N = 5000$

Are all nodes of the whole system **equally likely** to **exist in both layers?**

# **Nodes with high degree in IPv4 are more likely** to be present in the IPv6 network as well



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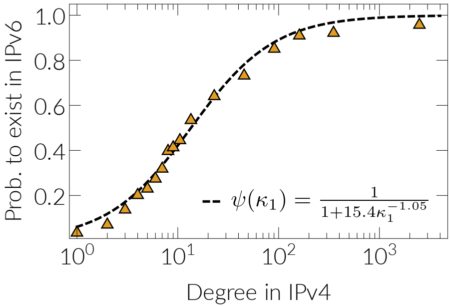
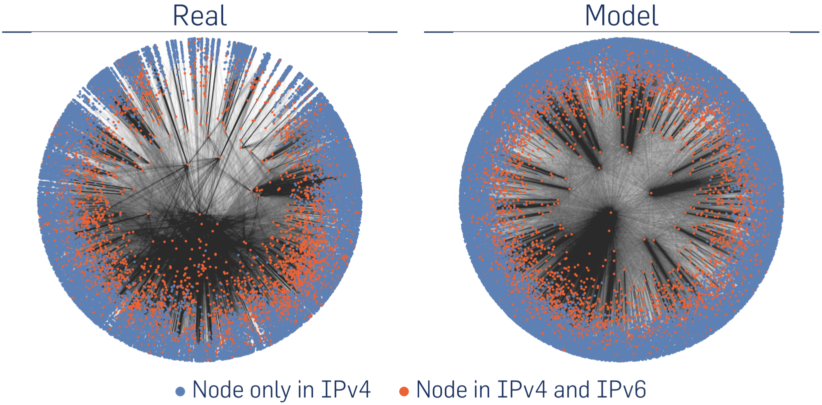



We select nodes from the IPv4 layer that also exist in IPv6 with **degree dependent probability.**

# **Internet multiplex model allows to study the performance** of mutual greedy routing for arbitrary correlations



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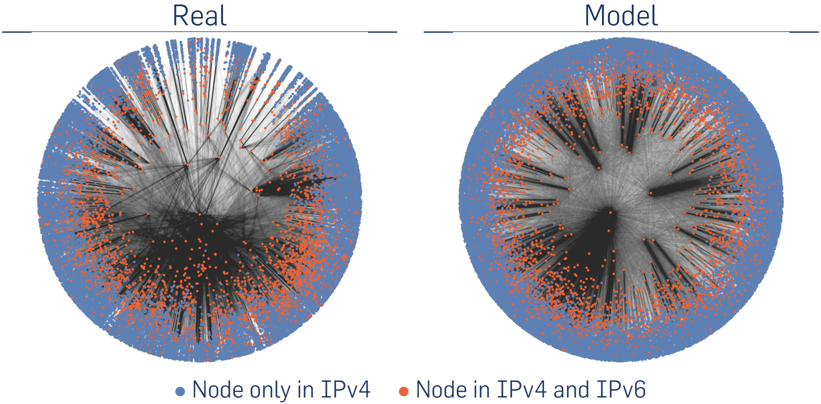


Can we **quantify the correlations** in the **real Internet** multiplex?

# **We can quantify the radial and angular correlations** present in the real IPv4 IPv6 Internet multiplex

**Radial correlations**

Person correlations coefficient between radial coordinates

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Match overlap from empirical and synthetic coordinates

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Person correlations coefficient between radial coordinates

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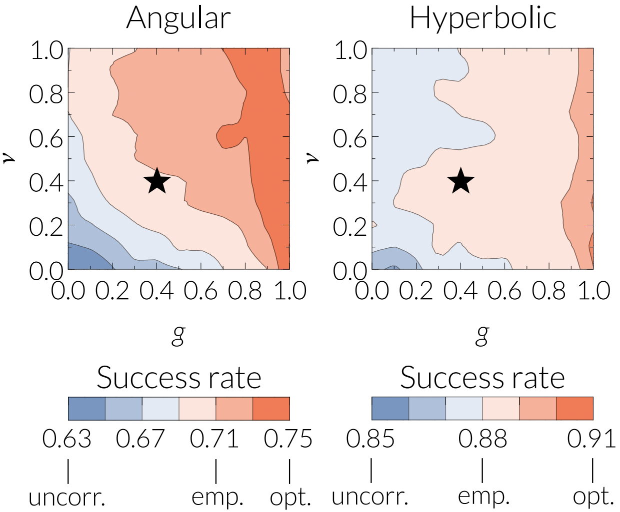
### Angular correlations

Match overlap from empirical and synthetic coordinates

$$g_E = 0.4$$

Do the **correlations** present in the **real Internet** help **navigation?**

# **Existing correlations in the real Internet multiplex increase** performance of mutual greedy routing significantly



# Geometric correlations in real Internet multiplex
can be measured and favor mutual greedy routing

**High degree**
nodes tend to exist
in both layers

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**Correlations**
in real Internet favor navigation

# Summary & outlook

# **Geometric correlations in real multiplex networks yield** a powerful framework for understanding these systems

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**...enable**
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prediction

**...are essential**
to improve mutual
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**uncover links**
between terrorists knowing another network

**improve search**
and navigation in decentralized systems

Kaj Kolja Kleineberg

Fragkiskos Papadopoulos

Maria Ángeles Serrano

Marián Boguñà

# Geometric correlations in real multiplex networks yield a powerful framework for understanding these systems

**Reference:**

K.-K. Kleineberg, M. Boguña, M.A. Serrano, F. Papadopoulos. arXiv:1601.04071, 2016

**Kaj Kolja Kleineberg:**


- kkl@ffn.ub.edu
- @KoljaKleineberg
- koljakleineberg.wordpress.com
- Kaj Kolja Kleineberg

Universitat de Barcelona

complexity lab barcelona

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## Kaj Kolja Kleineberg:


- kkl@ffn.ub.edu
- **@KoljaKleineberg ← Slides!**
- koljakleineberg.wordpress.com
- Kaj Kolja Kleineberg

Universitat de Barcelona

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# IMAGE CREDITS

Compass: Martin Fisch
Message in bottle: Susanne Nilsson
Old globe: jayneandd
Compass: Creative Stall
Compass (navigate): Creative Stall
Internet router: Thomas Uebe
Train: Naomi Atkinson
fly (drosophila): Daan Kauwenberg
worm (celegans): anbileru adaleru
bain network: parkjisun

coauthor: Matt Wasser
community: Edward Boatman
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